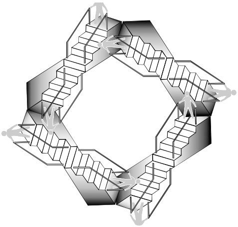
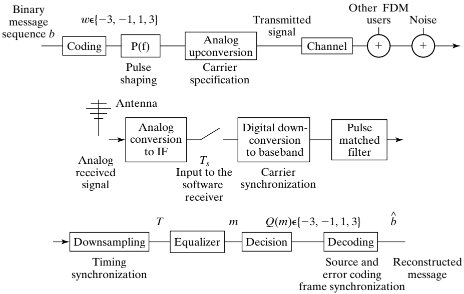
CARRIER RECOVERY

- Phase Tracking
  - Squared Difference
  - Phase-locked Loop
  - Costas Loop
  - Decision Directed
- Frequency Tracking



adaptive components

Carrier Phase Tracking



- A fixed phase offset between the transmitter and carrier oscillators results in an attenuation in the downconverted signal by the cosine of this phase difference.
- We seek algorithms for adjusting the receiver mixer's phase that can track (slow) time variations in the transmitter's phase.
- We treat carrier phase tracking as a single-parameter adaptation problem.

Adaptive Algorithm Development

Our (single-parameter) adaptive algorithm development strategy:

1. Propose a cost function assessing behavior over measured data set.
2. Check location of minima and maxima in terms of adjusted parameter to see if in desired location.
3. Pursue (small stepsize) gradient descent strategy (with its commutability of averaging and differentiation). The correction term must be calculable from available signals.
4. Test performance.

Phase Tracking

Carrier extraction

- For AM with large carrier, we could narrowly BPF the received signal to extract (mostly) just the carrier and then use a Fourier Transform or build a simpler sinusoid tracker that finds the carrier signal's phase.
- For AM with suppressed carrier we have to process the received upconverted signal

$$r(t) = s(t)\cos(2\pi f_c t + \phi)$$

which does not include an additive carrier.
- Consider squaring the received signal and using (A.4) to produce

$$r^2(t) = (1/2)s^2(t)[1 + \cos(4\pi f_c t + 2\phi)]$$

Phase Tracking (cont'd)

Carrier extraction (cont'd)

- Rewrite $s^2(t)$ as the sum of its (positive) average value and the variation about this average $s^2(t) = s^2_{avg} + v(t)$, so

$$\begin{aligned} r^2(t) = & \quad \frac{1}{2}s^2(t)[1 + \cos(4\pi f_c t + 2\phi)] \\ = & \quad (1/2)[s^2_{avg} + v(t) + s^2_{avg}\cos(4\pi f_c t + 2\phi) \\ & \quad +v(t)\cos(4\pi f_c t + 2\phi)] \end{aligned}$$

- A narrow bandpass filter centered at $2f_c$ with phase shift ρ at $2f_c$ extracts

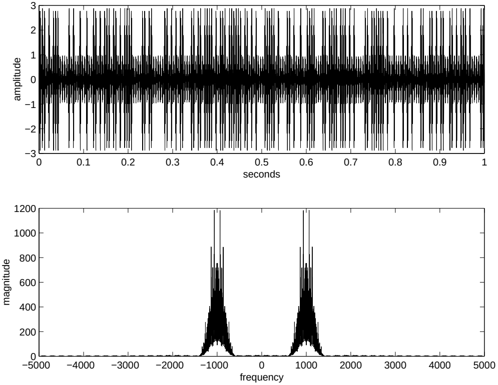
$$r_p(t) = (1/2)s^2_{avg}\cos(4\pi f_c t + 2\phi + \rho)$$

from r^2 while passing a bit of v about $2f_c$.

Phase Tracking (cont'd)

Carrier Extraction (cont'd)

For 1 second of a 4-PAM signal with symbol width $T = 0.005$, sample period (with an oversample factor of 50) $T_s = 0.0001$, and a carrier with frequency $f_c = 1000$ and phase $\phi = -1$, (from `pulrecsig`) the received signal and its spectrum are



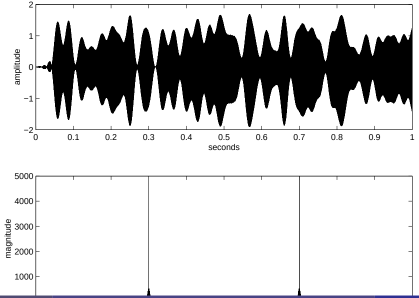
Phase Tracking (cont'd)

Carrier Extraction (cont'd)

Passing the received signal with $f_c = 1000$

$$r(kT_s) = s(kT_s)\cos(2\pi f_c kT_s + \phi)$$

through a squarer and a BPF centered at 2000 Hz with approximately 100 Hz passband and mod($\rho, 2\pi$)=0 (where mod(a, b) produces the remainder after division of a by b) yields r_p in time and frequency (from `pllpreprocess`)



Phase Tracking (cont'd)

Squared Difference

We have extracted from the received PAM signal a signal

$$r_p(kT_s) \approx (1/2)s_{avg}^2 \cos(4\pi f_c kT_s + 2\phi + \rho)$$

that crudely approximates a cosine with twice the frequency and phase of the carrier. Assume we generate a local sinusoid of frequency $f_0 = f_c$ and select θ in

$$\cos(4\pi f_0 kT_s + 2\theta + \mathrm{mod}(\rho, 2\pi))$$

to attempt to match r_p by minimizing

$$\begin{aligned} J_{SD} = & \quad \frac{1}{4P}\sum_{k=1}^{P}[\frac{2}{s_{avg}^2} r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta + \mathrm{mod}(\rho, 2\pi))]^2 \\ = & \quad \frac{1}{4}\mathrm{avg}\{[\frac{2}{s_{avg}^2} r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta + \mathrm{mod}(\rho, 2\pi))]^2\} \end{aligned}$$

which should happen at $\theta = \phi$ for a well-extracted cosine in r_p.

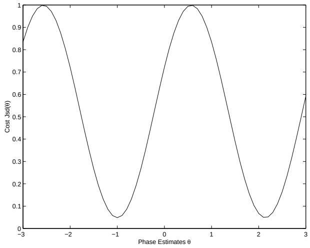
Phase Tracking (cont'd)

Squared Difference (cont'd)

The average squared difference cost

$$J_{SD} = \frac{1}{4}\text{avg}\{[\frac{2}{s_{avg}^2}r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta + \text{mod}(\rho, 2\pi))]^2\}$$

can be formed (for the previous example of carrier extraction via `pllpreprocess` where for our BPF, $\text{mod}(\rho, 2\pi) = 0$) for various fixed θ producing



This cost function has a minimum at the desired location of $\theta = \phi = -1$

Phase Tracking (cont'd)

Squared Difference (cont'd)

We will now analytically examine the average squared difference cost J_{SD} with $\psi = \text{mod}(\rho, 2\pi)$

$$J_{SD} = \frac{1}{4}\text{avg}\left\{\left(\frac{2}{s_{avg}^2}r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta + \psi)\right)^2\right\}$$

- First we presume that $(2/s_{avg}^2)r_p$ is very nearly the carrier at double frequency and with double phase plus BPF phase shift ρ with $\psi = \text{mod}(\rho, 2\pi)$

$$\cos(4\pi f_0 kT_s + 2\phi + \psi)$$

- Second, we note (see Appendix G) that the averaging operation $(\text{avg} \sim (1/P)\sum_{k=1}^{P})$ is a lowpass filter with impulse response $\{1/P, 1/P, ..., 1/P\}$.
- Thus,

$$J_{SD}(\theta) \approx \frac{1}{4}LPF\{(\cos(4\pi f_0 kT_s + 2\phi + \psi) - \cos(4\pi f_0 kT_s + 2\theta + \psi))^2\}$$

Phase Tracking (cont'd)

Squared Difference (cont'd)

- Expanding the square yields

$$\begin{aligned} J_{SD}(\theta) &\approx \frac{1}{2} LPF\Big\{\cos^2(4\pi f_0 kT_s + 2\phi + \psi) \\ &-2\cos(4\pi f_0 kT_s + 2\phi + \psi)\cos(4\pi f_0 kT_s + 2\theta + \psi) \\ &+ \cos^2(4\pi f_0 kT_s + 2\theta + \psi)\Big\} \end{aligned}$$

- Using (A.4) $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ and (A.9)

$$\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$$

the cost function becomes

$$\begin{aligned} J_{SD}(\theta) &\approx \frac{1}{8} LPF\Big\{2 + \cos(8\pi f_0 kT_s + 4\phi + 2\psi) \\ &-2\cos(2\phi - 2\theta) - 2\cos(8\pi f_0 kT_s + 2\phi + 2\theta + 2\psi) \\ &+ \cos(8\pi f_0 kT_s + 4\theta + 2\psi)\Big\} \end{aligned}$$

Phase Tracking (cont'd)

Squared Difference (cont'd)

- With the linearity of the LPF (so the LPF of the sum is the sum of the LPFs of each summand) and the cutoff frequency of the LPF less than $4f_0$ (which, for the averaging operation in J_{SD}, can be made lower for larger P)

$$J_{SD}(\theta) \approx \frac{1}{4}(1 - \cos(2\phi - 2\theta))$$

- For a fixed ϕ a full period ranging in amplitude from approximately 0 to 1 is traversed every π along the θ axis by the numerically generated cost function in agreement with this functional form of J_{SD}.

Phase Tracking (cont'd)

Squared Difference (cont'd)

- Our next step in our adaptive algorithm development strategy is to form the gradient of the cost
$$\frac{\partial}{\partial\theta}[\tfrac{1}{4}\text{avg}\{(\tfrac{2}{s_{avg}^2}r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta + \psi))^2\}]|_{\theta=\theta[k]}$$
- From Appendix G, we can commute the differentiation and averaging and perform the averaging operation with a LPF with cutoff less than $4f_0$
$$\sim \text{LPF}\{\tfrac{\partial}{\partial\theta}[\tfrac{1}{4}(\tfrac{2}{s_{avg}^2}r_p(kT_s) - \cos(4\pi f_0 kT_s + 2\theta + \psi))^2]|_{\theta=\theta[k]}\}$$
and retain an accurate approximation of the gradient for a small stepsize update.

Phase Tracking (cont'd)

Squared Difference (cont'd)

- Using (A.59) and (A.60), the partial derivative with respect to θ inside the average can be evaluated as

$$\left(\frac{2}{s_{avg}^2} r_p(kT_S) - \cos(4\pi f_0 kT_s + 2\theta[k] + \psi)\right) \cdot \sin(4\pi f_0 kT_s + 2\theta[k] + \psi)$$

- We will assume that $2/s_{avg}^2$ and ψ are known (or computed) at the receiver.
- The resulting *squared difference* carrier phase tracking algorithm is

$$\begin{aligned}\theta[k+1] &= \theta[k] - \mu \cdot \mathrm{LPF}\Big\{\Big(\frac{2}{s_{avg}^2} r_p(kT_s) \\ &\quad - \cos(4\pi f_0 kT_s + 2\theta[k] + \psi)\Big) \sin(4\pi f_0 kT_s + 2\theta[k] + \psi)\Big\}\end{aligned}$$

Phase Tracking (cont'd)

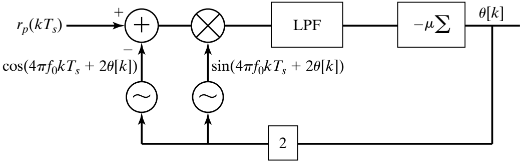
Squared Difference (cont'd)

- The signal r_p is the output of the squarer and narrow BPF combination with $\psi = \mathrm{mod}(\rho, 2\pi)$ where ρ is the phase of the preprocessing BPF at $2f_c$.
- In our example, the average s^2 can be calculated (in advance) from the average squared sample from a single pulse shape (which for `hamming(50)` is 0.3896) times the averaged squared source symbol (which is 5 for equally likely 4-PAM symbols of ± 1s and ± 3s) $\Rightarrow$ $s_{avg}^2 \approx 2 \Rightarrow 2/s_{avg}^2 \approx 1$.
- Or an AGC can be used to scale r_p to be a unit amplitude sinusoid.

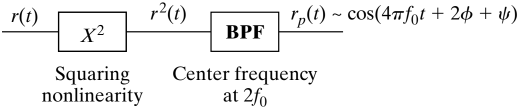
Phase Tracking (cont'd)

Squared Difference (cont'd)

Squared difference carrier recovery system:



where input r_p is the processed received signal of



and appropriate gain $(2/s_{avg}^2)$ has been implicitly included in BPF which has phase shift ψ at frequency $2f_c$.

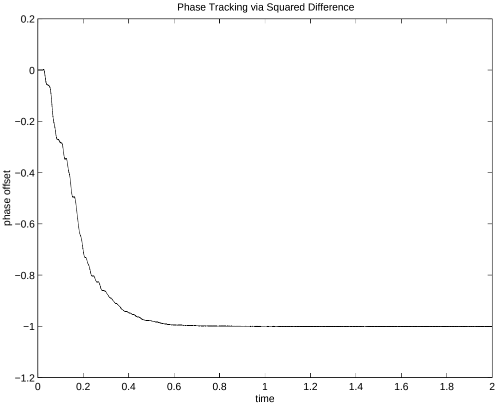
When ψ is nonzero, it should be added in carrier recovery system schematic after $2\theta[k]$ term in both oscillators.

Phase Tracking (cont'd)

Squared Difference (cont'd)

Sometimes (as in `pllsd`), the explicit LPF in the carrier recovery loop is removed and the LPF action of the integrator/summer block (Σ) suffices.

Phase acquisition example (from `pllsd` with $\mu = 0.001$)



Phase Tracking (cont'd)

Phase-locked Loop (PLL)

To introduce a phase-locked loop, the most widely known carrier recovery scheme, we present a candidate cost function producing the PLL.

- Reconsider the output of the squarer and narrow BPF, which is a scaled version of the carrier $r_p(kT_s) \approx g\cos(4\pi f_0 kT_s + 2\phi + \psi)$ where g is $s_{avg}^2/2$ times the square of the product of the channel and BPF gains at $2f_0$, and ψ is the BPF phase (mod 2π) at $2f_0$.
- Consider downconverting $r_p(kT_s)$ with our (unsynchronized) receiver oscillator's output and form

$$
\begin{aligned}
&r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)\\
&\qquad \approx g\cos(4\pi f_0 kT_s + 2\phi + \psi)\cos(4\pi f_0 kT_s + 2\theta + \psi)\\
&\qquad = \frac{g}{2}\{\cos(2\phi - 2\theta) + \cos(8\pi f_0 kT_s + 2\phi + 2\theta + 2\psi)\}
\end{aligned}
$$

Phase Tracking (cont'd)

Phase-locked Loop (PLL)

- Lowpass filtering (half of) this product with a LPF with cutoff below $4f_0$ produces

$$\mathrm{LPF}\{\frac{1}{2}r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)\} \approx \frac{g}{4}\cos(2\phi - 2\theta)$$

  which is maximized when $2\phi - 2\theta = 2n\pi \Rightarrow \phi - \theta = n\pi$.

- Value of positive, finite g does not effect locations of max/minima.
- We will choose to maximize

$$J_{PLL} = \frac{1}{P}\sum_{k=k_0}^{k_0+P}\{r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)\}$$

$$= \mathrm{avg}\{r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)\}$$

$$\sim \mathrm{LPF}\{r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)\}$$

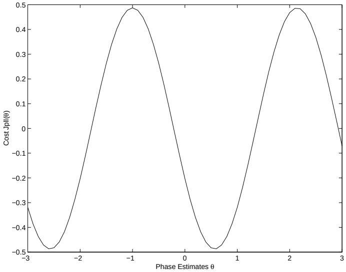
Phase Tracking (cont'd)

PLL (cont'd)

As a numerical test for extrema, the PLL cost

$$J_{PLL} = \mathrm{LPF}\{r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)\}$$

can be formed for various fixed θ producing (via `pllconverge`)



A maximum (near 0.5 with $g \approx 1$ in this case) appears at the desired location of $\theta = \phi = -1$ (with $\psi = 0$) and at locations an integer multiple of π away, as predicted in the preceding analysis.

Phase Tracking (cont'd)

PLL (cont'd)

Following a gradient ascent strategy for maximization, compose

$$\theta[k+1] = \theta[k] + \bar{\mu}\frac{\partial}{\partial\theta}[\text{avg}\{r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)\}]|_{\theta=\theta[k]}$$

With a small stepsize assuring (approximate) commutability of differentiation and average

$$\theta[k+1] = \theta[k] + \bar{\mu}\cdot\text{avg}\{\frac{\partial}{\partial\theta}[r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)]|_{\theta=\theta[k]}\}$$

where

$$\begin{aligned}&\frac{\partial}{\partial\theta}[r_p(kT_s)\cos(4\pi f_0 kT_s + 2\theta + \psi)]|_{\theta=\theta[k]}\\ &= -2r_p(kT_s)\sin(4\pi f_0 kT_s + 2\theta[k] + \psi)\end{aligned}$$

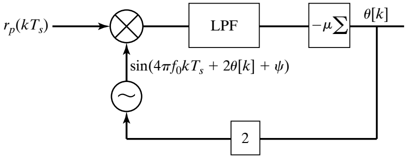
This produces

$$\theta[k+1] = \theta[k] - \mu\text{LPF}\{r_p(kT_s)\sin(4\pi f_0 kT_s + 2\theta[k] + \psi)\}$$

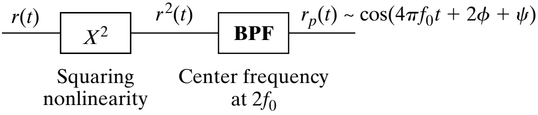
Phase Tracking (cont'd)

PLL (cont'd)

PLL carrier recovery system:



where input r_p is the processed received signal of



and appropriate gain $(2/s_{avg}^2)$ has been implicitly included (unnecessarily for this PLL, unlike the squared difference scheme) in BPF which has phase shift ψ at frequency $2f_0$.

When ψ is nonzero, it should be added in carrier recovery system schematic after $2\theta[k]$ term in the oscillator.

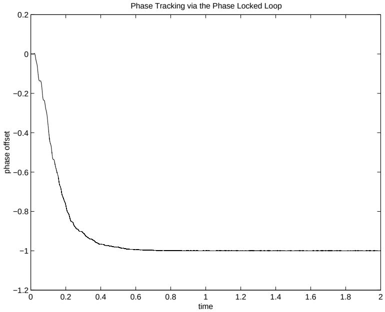
Phase Tracking (cont'd)

PLL (cont'd)

For the PLL algorithm with explicit LPF preceding integrator/summer removed

$$\theta[k+1] = \theta[k] - \mu r_p(kT_s) \sin(4\pi f_0 kT_s + 2\theta[k] + \psi)$$

a typical learning curve (from `pllconverge`) for a stepsize of $\mu = 0.001$ for our continuing example (with $\psi = 0$ and an objective of $\theta = -1$) is



Phase Tracking (cont'd)

Costas Loop

Now, we seek an algorithm not based on a presumption of carrier extraction from the received signal.

- Reconsider the received signal

$$r(kT_s) = s(kT_s)\cos(2\pi f_c kT_s + \phi).$$

- Assume the transmitter carrier frequency f_c and receiver frequency are identical ($f_c = f_0$) and form

$$2r(kT_s)\cos(2\pi f_0 kT_s + \theta)$$
$$= s(kT_s)[\cos(\phi - \theta) + \cos(4\pi f_0 kT_s + \phi + \theta)]$$

- With a LPF cutoff below $2f_0$

$$\text{LPF}\{2r(kT_s)\cos(2\pi f_0 kT_s + \theta)\} = v(kT_s)\cos(\phi - \theta)$$

where $v(kT_s) = \text{LPF}\{s(kT_s)\}$. If the cutoff frequency of the LPF is above the bandwidth of the baseband waveform s, then v is s.

Phase Tracking (cont'd)

Costas Loop (cont'd)

- As a cost function, consider

$$\begin{aligned} J_C(\theta) &= \frac{1}{P} \sum_{k=k_0-(P-1)}^{k_0} \{\mathrm{LPF}[2r(kT_s)\cos(2\pi f_0 kT_s + \theta)]\}^2 \\ &\approx \operatorname{avg}\{v^2(kT_s)\cos^2(\phi-\theta)\} \end{aligned}$$

- Because the squared cosine term is fixed, given (A.4)

$$\begin{aligned} &\operatorname{avg}\{v^2(kT_s)\cos^2(\phi-\theta)\} \\ &= \left(\operatorname{avg}\{v^2(kT_s)\}\right)\frac{(1+\cos(2(\phi-\theta)))}{2} \end{aligned}$$

  and assuming that the average of v^2 is fixed, this cost function will be maximized with a value equal to the average of v^2 (which is average value of $\{\mathrm{LPF}[s]\}^2$) at $\phi - \theta = \pi n$ or $\theta = \phi + \pi n$ for all (positive and negative) integers n.

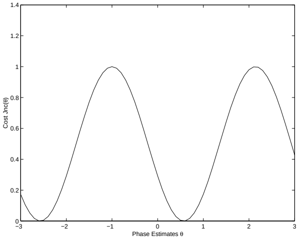
Phase Tracking (cont'd)

Costas Loop (cont'd)

We can numerically check the extrema of a normalized J_C as

$$J_{NC}(\theta) = \frac{\frac{1}{P}\sum_{k=1}^{P}\left(\mathrm{LPF}\{2r(kT_s)\cos(2\pi f_0 kT_s + \theta)\}\right)^2}{\frac{1}{P}\sum_{k=1}^{P}\left(\mathrm{LPF}\{s(kT_s)\}\right)^2}$$

where r is the received signal for our continuing example for various fixed θ producing



This normalized cost function matches $(1 + \cos(2(\phi - \theta)))/2$, as anticipated.

Phase Tracking (cont'd)

Costas Loop (cont'd)

Our next step in our algorithm creation strategy is to interchange the averaging and differentiation in the gradient ascent update

$$\theta[k+1] = \theta[k] + \bar{\mu}\frac{\partial}{\partial\theta}[\text{avg}\{(\text{LPF}\{2r(kT_s)\cdot\cos(2\pi f_0 kT_s+\theta)\})^2\}]|_{\theta=\theta[k]}$$

With

$$\text{LPF}\{2r(kT_s)\cos(2\pi f_0 kT_s+\theta)\} = v(kT_s)\cos(\phi-\theta)$$

the update can be written as

$$\begin{aligned}\theta[k+1] &= \theta[k] + \bar{\mu}\cdot\text{avg}\{\frac{\partial}{\partial\theta}[v^2(kT_s)\cos^2(\phi-\theta)]|_{\theta=\theta[k]}\} \\ &= \theta[k] + \mu\cdot\text{avg}\{v^2(kT_s)(\cos(\phi-\theta)\frac{\partial\cos(\phi-\theta)}{\partial\theta})|_{\theta=\theta[k]}\}\end{aligned}$$

and from (A.62) we wish to form

$$\theta[k+1] = \theta[k] + \mu\cdot\text{avg}\{v^2(kT_s)\cos(\phi-\theta[k])\sin(\phi-\theta[k])\}$$

Phase Tracking (cont'd)

Costas Loop (cont'd)

- Given

$$\mathrm{LPF}\{2r(kT_s)\cos(2\pi f_0 kT_s + \theta)\} = v(kT_s)\cos(\phi - \theta)$$

  to compose the update from measurable signals we need to find a realizable expression for $v(kT_s)\sin(\phi - \theta)$.

- For a LPF with cutoff under $2f_0$, defining $v = \mathrm{LPF}\{s\}$ and using (A.10) and (A.11) produces
  $\mathrm{LPF}\{2r(kT_s)\sin(2\pi f_0 kT_s + \theta)\}$

$$= \mathrm{LPF}\{s(kT_s)\cos(2\pi f_0 kT_s + \phi)\sin(2\pi f_0 kT_s + \theta)\}$$

$$= \mathrm{LPF}\{s(kT_s)(\sin(\theta - \phi) - \sin(4\pi f_0 kT_s + \phi + \theta))\}$$

$$= -v(kT_s)\sin(\phi - \theta)$$

Phase Tracking (cont'd)

Costas Loop (cont'd)

Thus, a small stepsize gradient ascent algorithm (for maximization of J_C) is

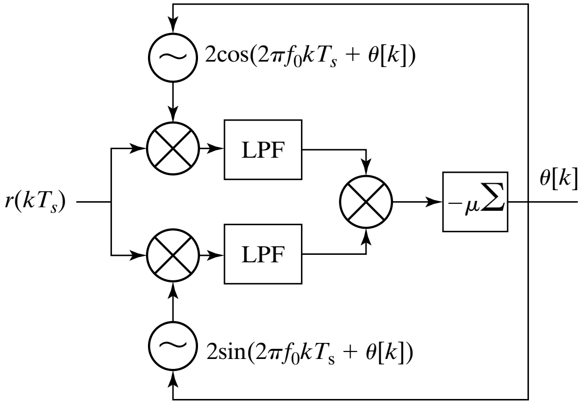
$$\theta[k+1] = \theta[k] - \mu \cdot \operatorname{avg}\Big[\mathrm{LPF}\{2r(kT_s)\cos(2\pi f_0 kT_s + \theta[k])\}$$
$$\cdot \mathrm{LPF}\{2r(kT_s)\sin(2\pi f_0 kT_s + \theta[k])\}\Big]$$

- The use of lowpass filtering in the update is predicated on a presumption that the LPF output is characterized by its asymptotic response.
- This effectively presumes $\theta[k]$ remains fixed for a sufficiently long time for this asymptotic behavior to be achieved.
- We rely on a small stepsize μ to keep $\theta[k]$ variations modest in the (relatively) short time frame anticipated for LPF achievement of asymptotic behavior.

Phase Tracking (cont'd)

Costas Loop (cont'd)

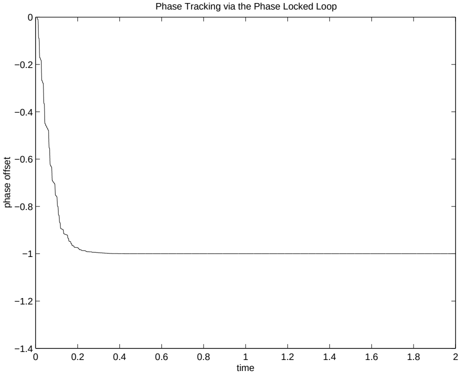
Schematic for Costas loop carrier phase recovery with the "outer" averaging removed (which presumes that the integrator/summer of the update will provide sufficient averaging):



Phase Tracking (cont'd)

Costas Loop (cont'd)

A typical learning curve for this Costas loop carrier phase recovery scheme (as shown in the preceding schematic without explicit averaging in the update) on our continuing example (with an objective of -1) is (from `costasloop` with a stepsize of $\mu = 0.001$)



Phase Tracking

Decision directed

- Errors in carrier phase recovery will be reflected in errors in soft decisions x.
- Assuming correct hard decisions, the decision-directed cost

$$J_{DD} = \frac{1}{P} \sum_{k=1}^{P} (Q\{x(kT)\} - x(kT))^2$$

$$= \text{avg}\{(Q\{x(kT)\} - x(kT))^2\}$$

can be formed for various fixed θ, following complete demodulation via digital mixing with a fixed receiver oscillator phase θ producing

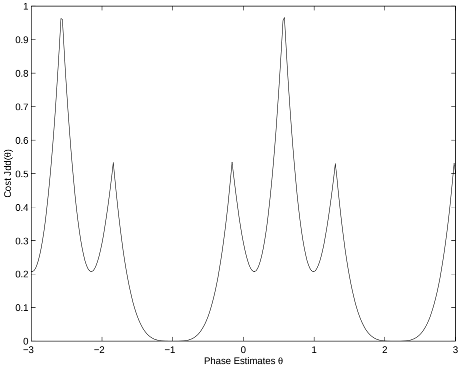
$$x(kT_s) = \text{LPF}\{2r(kT_s)\cos(2\pi f_0 kT_s + \theta)\}$$

and downsampling by a factor of M (assuming $T = MT_s$ and selection of desired baud-timing setting) producing T-spaced soft symbol decisions.

Phase Tracking

Decision directed (cont'd)

For our continuing example, we can numerically evaluate J_{DD} over an $\{x\}$ dataset for various θ and compose (in `ddcrt`)



Phase Tracking (cont'd)

Decision directed (cont'd)

- Maxima of the decision-directed cost function occur in "worst" case when $x \approx 0$ but $|\mathrm{Q}\{x\}| = 1$ so $(\mathrm{Q}\{x\} - x)^2 \approx 1$.
- The decision-directed cost function has a minimum at the desired location of $\theta = \phi = -1$ and at locations an integer multiple of π away.
- J_{DD} also has other local minima making initialization critical to achieving the desired system behavior.
- Accordingly, a decision-directed cost (and associated adaptation algorithm) is often used only to maintain lock and provide tracking with low algorithmic complexity once initial carrier acquisition has occured.

Phase Tracking

Decision directed (cont'd)

Now we examine the approximation of the gradient of the decision-directed cost function by evaluating the gradient of J_{DD} after swapping average and differentiation (see Appendix G) under a small stepsize presumption

- Because $\partial[\mathrm{Q}\{x\}]/\partial x = 0$ almost everywhere

$$\begin{aligned} \frac{\partial J_{DD}}{\partial \theta} &\approx \operatorname{avg}\left\{ \frac{\partial (Q\{x(kT)\} - x(kT))^2}{\partial \theta} \right\} \\ &\approx -2\operatorname{avg}\left\{ (\mathrm{Q}\{x(kT)\} - x(kT)) \frac{\partial x(kT)}{\partial \theta} \right\} \end{aligned}$$

- For a basic mixer downconversion

$$\begin{aligned} x(iT) &= x(kT_s)|_{k=Mi} \\ &= 2[\mathrm{LPF}\{r(kT_s)\cos(2\pi f_0 kT_s + \theta)\}]|_{k=Mi} \end{aligned}$$

where $\quad r(kT_s) = s(kT_s)\cos(2\pi f_0 kT_s + \phi)$

which we can use to form $\partial x / \partial \theta$.

Phase Tracking

Decision directed (cont'd)

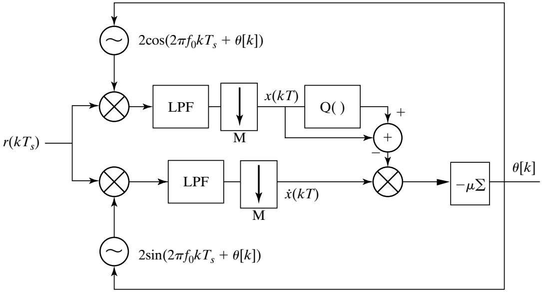
- Swapping the order of LPF and differentiation (see Appendix G) in $\partial x/\partial\theta$ and using (A.61) and (A.62) yields $\frac{\partial(Q\{x(kT_s)\}-x(kT_s))^2}{\partial\theta} \approx$ $4(\mathrm{Q}\{x[k]\} - x[k]) \cdot (\mathrm{LPF}\{r(kT_s)\sin(2\pi f_0 kT_s + \theta[k])\})$
- So, the decision-directed carrier phase tracking update is

$$\theta[k+1] = \theta[k] - \mu(\mathrm{Q}\{x[k]\} - x[k]) \cdot \mathrm{LPF}\{r(kT_s)\sin(2\pi f_0 kT_s + \theta[k])\}$$

Phase Tracking (cont'd)

Decision directed (cont'd)

The decision-directed carrier recovery schematic:



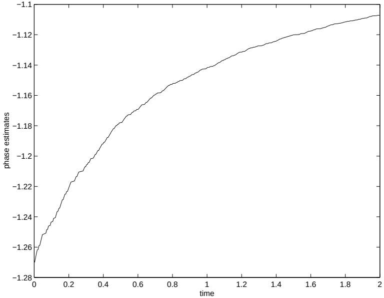
- The downsamplers are presumed coordinated and synchronized for symbol recovery.
- Updating of θ only occurs once per symbol, rather than once per sample with $i = kM$.

Phase Tracking (cont'd)

Decision directed (cont'd)

Adapted θ trajectories (from `pllddd`) for two starting points:

- $\theta[1] = -1.27$ and $\mu = 0.03$

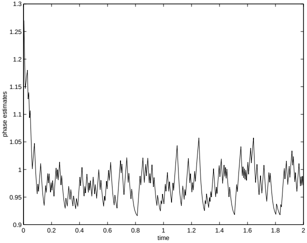


- Slow convergence here is related to flatness of cost function in vicinity of global minimum.

Phase Tracking (cont'd)

Decision directed (cont'd)

- $\theta[1] = 1.27$ and $\mu = 0.03$



- Rapid convergence with this initialization is due to steepness of cost function in vicinity of initialization.
- Substantial asymptotic rattling is due to nonzero cost at acquired local minimum.

Frequency Tracking

Consider the situation where both the receiver oscillator's frequency and phase are off.

- Received signal

$$r(kT_s) = s(kT_s)\cos(2\pi f_c kT_s + \phi)$$

- Receiver mixer LPF output assuming $f_0 - f_c$ within LPF passband

$$y(kT_s) = \mathrm{LPF}\{2r(kT_s)\cos(2\pi f_0 kT_s + \theta)\}$$
$$= s(kT_s)\cos(2\pi(f_c - f_0)kT_s + \phi - \theta)$$

- With θ adaptively adjusted as $\theta[k]$, perfect carrier recovery occurs only if

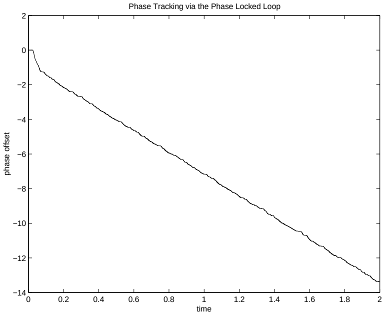
$$\theta[k] = 2\pi(f_c - f_0)kT_s + \phi$$

Frequency Tracking (cont'd)

- Single PLL produces

$$\theta[k] \rightarrow 2\pi(f_c - f_0)kT_s + \beta$$

- With $\phi = -1$, $f_c = 1000$, $f_0 = 1001$, and $\theta[1] = 0$, PLL (from `pllconverge`) with a stepsize of 0.005 produces

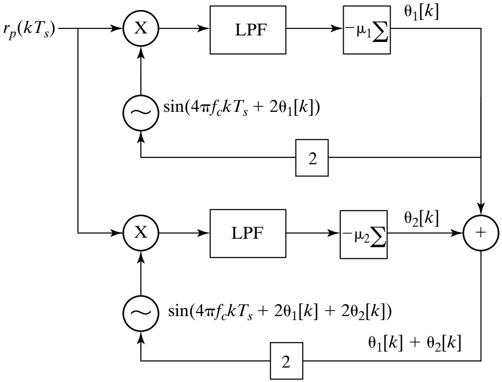


Frequency Tracking (cont'd)

- From plot take $\theta(4000) = -3.4009$ and $\theta(16000) = -10.9304$ to check
$$2\pi(f_c - f_0)kT_s + \beta = \theta[k]$$
- For $k = 4000$, $f_c - f_0 = -1$, and $T_s = .0001$ we achieve fair agreement:
$$-2.5133 + \beta = -3.4009 \Rightarrow \beta \approx -0.8876$$
$$-10.0531 + \beta = -10.9304 \Rightarrow \beta \approx -0.8773$$
- With $\phi = -1$, this means that single PLL is short by $\approx -1 - (-0.88) \approx -0.12$ between $k = 4000$ and $k = 16,000$ and beyond.
- Adding the phase estimate output of a first PLL to the feedback in a second, allows the second loop to track the remaining constant phase offset in matching the carrier phase.

Frequency Tracking (cont'd)

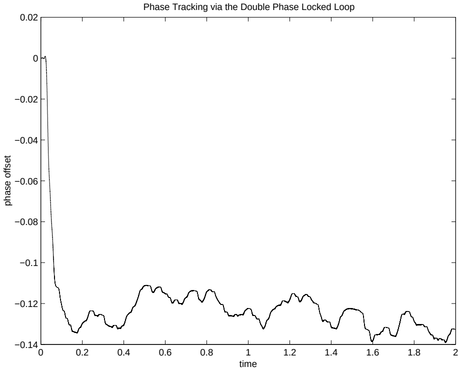
Dual-PLL schematic



where input r_p is the received signal passed through squarer and BPF with appropriate gain $(2/s^2_{avg})$ and phase shift ψ at frequency $2f_0$.
When ψ is nonzero, it should be added in carrier recovery system schematic after last $2\theta_i[k]$ term in both oscillators.

Frequency Tracking (cont'd)

Second PLL (from `pllconverge`) with stepsize of 0.0005 (which is a factor of 10 smaller than stepsize of first ramp-tracking stage) produces



with second loop removing remainder of ≈ -0.12.

Frequency Tracking

- Could try squared difference approach to minimize average squared difference between extracted carrier and reconstructed carrier with estimated frequency.
- Resulting squared difference cost function across frequency estimate as independent variable is flat with one deep well $\Rightarrow$ not a good surface for a gradient descent search.
- Can add a second integrator to single PLL feedback loop as alternative to second PLL.
- Can use other phase trackers in dual configuration.

NEXT... We examine the pulse shape and receive filters that aid the transition to (and from) an analog transmitted signal from (and to) a digital message sequence.